Written Exam at the Department of Economics winter 2016-17

Financial Econometrics A

Final Exam

Date: February 10th, 2017

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language for which you registered during exam registration.

This exam question consists of 6 pages in total

NB: If you fall ill during the actual examination at Peter Bangsvej, you must contact an invigilator in order to be registered as having fallen ill. Then you submit a blank exam paper and leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Please note there are a total of **9** questions that you should provide answers to. That is, **4** questions under *Question A*, and **5** under *Question B*.

Question A:

Consider the following DAR-X model given by

$$x_t = \phi x_{t-1} + \varepsilon_t, \tag{A.1}$$

where the error term ε_t satisfies

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim i.i.d.N(0,1) \tag{A.2}$$
$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta y_{t-1}^2. \tag{A.3}$$

Here y_t is some observed exogenous variable satisfying

$$y_t = \rho y_{t-1} + \eta_t, \quad \eta_t \sim i.i.d.N(0,1).$$

The model parameters $\theta = (\phi, \omega, \alpha, \beta, \rho)$ satisfy $\phi, \rho \in \mathbb{R}$, $\omega > 0$, and $\alpha, \beta \geq 0$. We assume that the processes (z_t) and (η_t) are independent.

Question A.1: Suppose that $\alpha = \beta = 0$. Under what conditions is x_t weakly mixing?

Question A.2: Consider the joint process $W_t = (x_t, y_t)$. Argue that W_t is a Markov chain.
It holds that the conditional density of W_t is

$$\begin{aligned} f(W_t|W_{t-1}) &= f(x_t, y_t|x_{t-1}, y_{t-1}) \\ &= f(x_t|x_{t-1}, y_{t-1})f(y_t|y_{t-1}). \end{aligned}$$

Derive expressions for the conditional densities $f(x_t|x_{t-1}, y_{t-1})$ and $f(y_t|y_{t-1})$, and argue that $f(x_t, y_t|x_{t-1}, y_{t-1})$ is positive and continuous in (x_t, y_t) and (x_{t-1}, y_{t-1}).
Explain briefly what this insight can be used for.

It can be shown (but do not do so) that the Markov chain W_t satisfies the drift criterion with drift function $\delta(W_t) = 1 + \|W_t\|^2 = 1 + x_t^2 + y_t^2$ if $\max(\rho^2 + \beta, \phi^2 + \alpha) < 1$.

Question A.3: Consider the OLS estimator for ϕ given by

$$\hat{\phi} = \frac{\sum_{t=1}^T x_{t-1}x_t}{\sum_{t=1}^T x_{t-1}^2}.$$

It holds that

$$\hat{\phi} - \phi = \frac{\sum_{t=1}^T x_{t-1}\varepsilon_t}{\sum_{t=1}^T x_{t-1}^2}.$$

Assume that $W_t = (x_t, y_t)$ is weakly mixing and satisfies the drift criterion such that $E[x_t^4] < \infty$ and $E[y_t^4] < \infty$.
Show that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T x_{t-1}\varepsilon_t \xrightarrow{D} N(0, v)$$

with $v = E[\omega x_{t-1}^2 + \alpha x_{t-1}^4 + \beta y_{t-1}^2 x_{t-1}^2]$.
Explain briefly what this property can be used for.

Question A.4: Instead of estimating only ϕ based on OLS, we may estimate all parameters $\theta = (\phi, \omega, \alpha, \beta, \rho)$ based on maximum likelihood estimation. For the model (A.1)-(A.3), the one-period VaR at risk level κ, $\mathrm{VaR}_{T,1}^{\kappa}$, is

$$\mathrm{VaR}_{T,1}^{\kappa} = -\phi x_T - \sigma_{T+1}\Phi^{-1}(\kappa), \quad \kappa \in (0, 1),$$

where $\Phi^{-1}(\cdot)$ denotes the inverse cdf of the standard normal distribution.
Explain briefly how you would compute an estimate of $\mathrm{VaR}_{T,1}^{\kappa}$.

Question B:

Suppose that the logarithm of the price of a share of stock is given by

$$p(t) = p(0) + \mu t + \sigma W(t), \quad t \in [0, T], \tag{B.1}$$

where $p(0) \in \mathbb{R}$ is some fixed initial value, $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, and $W(t)$ is a Brownian motion.

Recall here that the Brownian motion $W(t)$ has the properties

1. $W(0) = 0$.
2. W has independent increments, i.e. if $0 \leq r < s \leq t < u$, then

   $$W(u) - W(t) \text{ and } W(s) - W(r)$$

   are independent.
3. The increments are normally distributed, i.e.

   $$W(t) - W(s) \sim N(0, t - s)$$

   for all $0 \leq s \leq t$.

Suppose that we have observed the price $p(t)$ at $n + 1$ equidistant points

$$0 = t_0 < t_1 < \ldots < t_n = T,$$

with

$$t_i = \frac{i}{n}T, \quad i = 0, ..., n.$$

Based on these points we obtain n log-returns given by

$$r(t_i) = p(t_i) - p(t_{i-1}), \quad i = 1, ..., n.$$

Question B.1: Argue that $r(t_i)$ is normally distributed, i.e. show that

$$r(t_i) \sim N\left(\mu\frac{T}{n}, \sigma^2\frac{T}{n}\right).$$

Show that

$$\text{cov}(r(t_i), r(t_{i-1})) = 0.$$

Question B.2: We now seek to estimate the model parameters (μ, σ^2) based on maximum likelihood. Given the n log-returns, the log-likelihood function is (up to a constant and a scaling factor)

$$L_n(\mu, \sigma^2) = \sum_{i=1}^{n} \left\{ -\log(\sigma^2 \frac{T}{n}) - \frac{\left[r(t_i) - \mu \frac{T}{n}\right]^2}{\sigma^2 \frac{T}{n}} \right\}.$$

Let $\hat{\mu}$ denote the maximum likelihood estimator of μ.
Show that

$$\hat{\mu} = \frac{1}{T} \sum_{i=1}^{n} r(t_i) = \frac{1}{T} \left[p(T) - p(0)\right].$$

Argue that the sampling frequency of the log-returns over the interval $[0, T]$ does not have any influence on the estimate of μ.

Question B.3: Let $\hat{\mu}$ denote the maximum likelihood estimator derived in Question B.2.
Show that $\hat{\mu}$ is an unbiased estimator for μ, i.e. show that

$$E[\hat{\mu}] = \mu.$$

Moreover, show that the variance of the estimator is

$$\text{Var}(\hat{\mu}) = \frac{\sigma^2}{T}.$$

It can be shown (but do not do so) that these two properties ensure that $\hat{\mu} \xrightarrow{p} \mu$ as $T \to \infty$.

Question B.4: Assume now that $T = 1$, such that we have n observations of the log-returns over the time interval $[0, 1]$, which you may think of as the time interval over one trading day. Then the maximum likelihood estimator for σ^2 is given by

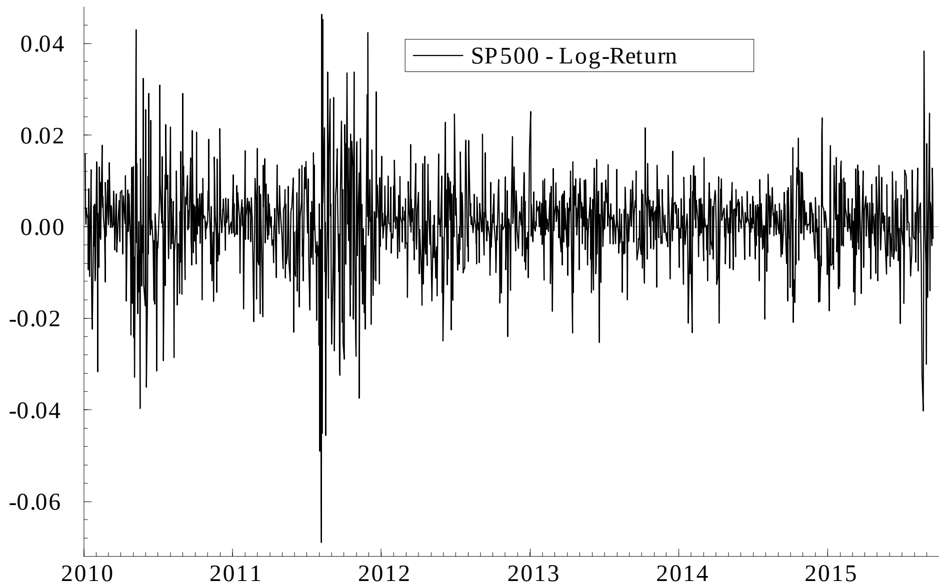
$$\hat{\sigma}^2 = \sum_{i=1}^{n} \left[r(t_i) - \frac{1}{n} \sum_{i=1}^{n} r(t_i) \right]^2.$$

Use that $r(t_i) = \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_i$, with $z_i \sim i.i.d. N(0, 1)$ in order to show that

$$\frac{1}{n} \sum_{i=1}^{n} r(t_i) \xrightarrow{p} 0 \quad \text{as } n \to \infty.$$

Explain briefly how $\hat{\sigma}^2$ is related to the Realized Volatility.

Question B.5: The following figure shows the daily log-returns of the S&P 500 index for the period January 4, 2010 to September 17, 2015.



Discuss briefly whether the model in (B.1) is a reasonable model for the daily log returns of the S&P 500 index.